Gordian Distance and Complete Alexander Neighbors

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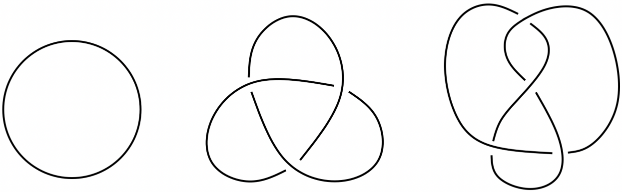
Knot Theory

Definition

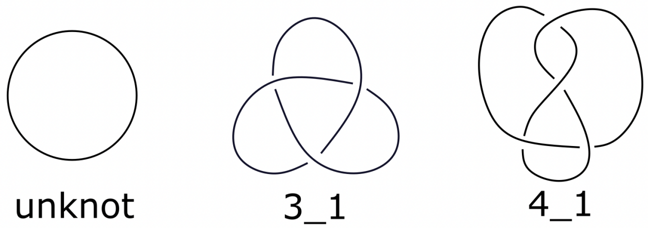
A **knot** is a smooth embedding of the circle S^1 in the 3-sphere S^3 considered up to ambient isotopy.



Knot Invariants

Definition

The **crossing number** of a knot K is the minimal number of crossings in any diagram of K.

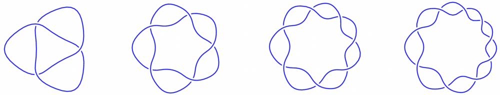


Knot Invariants

Definition

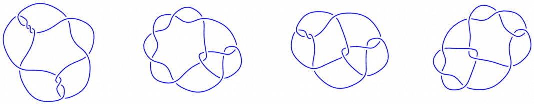
The **unknotting number** of a knot K is the minimal number of crossing changes required to transform K into the unknot.

The unknotting number of a $(p, q)-$ torus knot is $\frac{(p-1)(q-1)}{2}$ so there exist knots with arbitrarily large unknotting number.



3_1 (2,3)-torus knot | 5_1 (2,5)-torus knot | 7_1 (2,7)-torus knot | 9_1 (2,9)-torus knot

The unknotting number is unknown for many small knots.

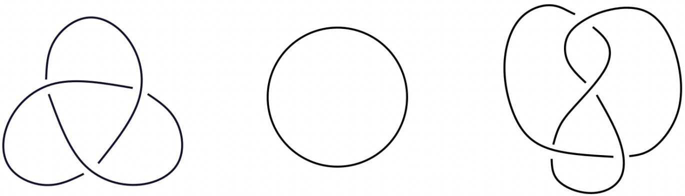


10_11 | 10_47 | 10_51 | 10_54

Knot Invariants

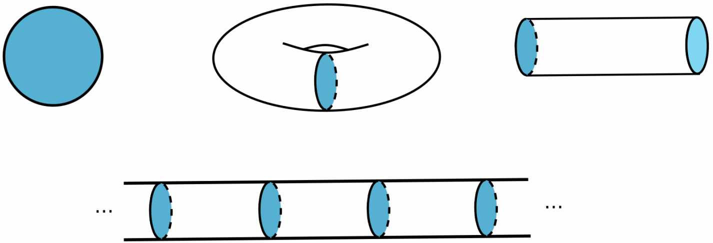
Definition

The **Gordian distance** between two knots K and K' is the minimal number of crossing changes necessary to go from a diagram of K to a diagram of K'.



Building an Infinite Cyclic Cover

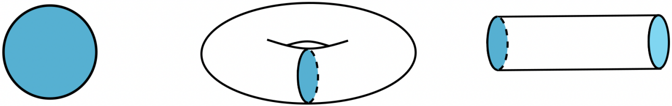
Let K be a knot in S^3. We can build the infinite cyclic cover of $S^3 \setminus K$ by cutting along an orientable 2-manifold with boundary $K \cong S^1$ (a Seifert surface of K) in S^3.



The Alexander Polynomial

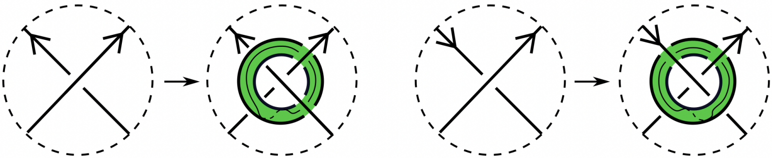
Definition

The **Alexander polynomial** $\triangle_K(t)$ of a knot K is the determinant of a presentation matrix (known as an **Alexander matrix**) for $H_1(X_\infty)$ as a module over $\mathbb{Z}[t, t^{-1}]$ where t is a covering transformation along the infinite cyclic cover X_∞ of $S^3 \setminus K$ from one lift of the complement of a Seifert surface of K in S^3 to the next adjacent lift.

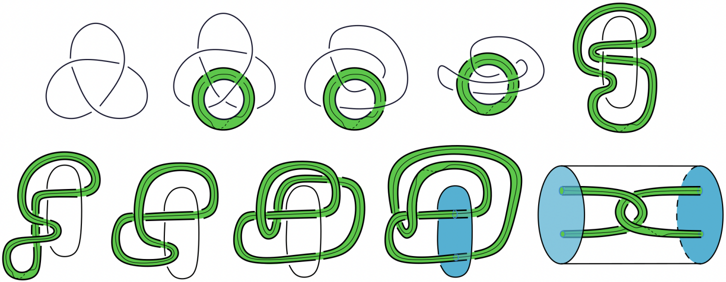


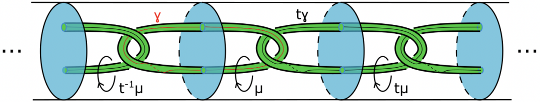
Dehn Surgery

K can be transformed into the unknot with a series of n crossing changes where n is the unknotting number of K.



The Alexander Polynomial of the Trefoil

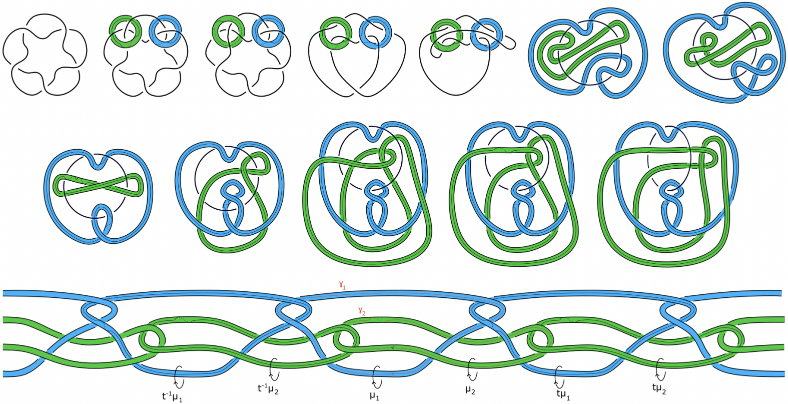




$$\gamma = t\mu - \mu + t^{-1}\mu = (t - 1 + t^{-1})\mu$$

$$\triangle_{3_1}(t) = \det(t - 1 + t^{-1}) = t - 1 + t^{-1}$$

The Alexander Polynomial of 5_1



$\gamma_1 = (t - 1 + t^{-1})\mu_1 + (-t^{-1} + 1)\mu_2 \qquad \gamma_2 = (-t + 1)\mu_1 + (t - 1 + t^{-1})\mu_2$

$$\triangle_{5_1}(t) = \det\begin{pmatrix} t - 1 + t^{-1} & -t^{-1} + 1 \\ -t + 1 & t - 1 + t^{-1} \end{pmatrix} = t^2 - t + 1 - t^{-1} + t^2$$

Alexander Polynomial

Definition

The **determinant** $\det(K)$ of a knot K is $|\triangle_K(-1)|$.

Definition

The **algebraic unknotting number** of a knot K is the minimal number of crossing changes required to transform K into a knot with trivial Alexander polynomial.

Characterization of the Alexander Polynomial

Every Alexander polynomial can be written as $p(t) \in \mathbb{Z}[t, t^{-1}]$ such that

1. $p(1) = \pm 1$ and
2. $p(t^{-1}) = p(t)$.

Conversely, every such polynomial is the Alexander polynomial of some knot.

K	$\triangle_K(t)$
	1
3_1 	$t - 1 + t^{-1}$
4_1 	$t - 3 + t^{-1}$
5_1 	$t^2 - t + 1 - t^{-1} + t^{-2}$
5_2 	$2t - 3 + 2t^{-1}$

Gordian Distance and the Alexander Polynomial

Theorem (Kondo 1978, [2])

For any Alexander polynomial $p(t)$, there exists a knot K with unknotting number one such that $\triangle_K(t) = p(t)$.

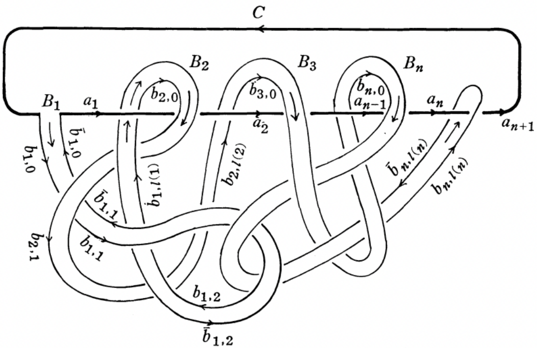


Fig. 4

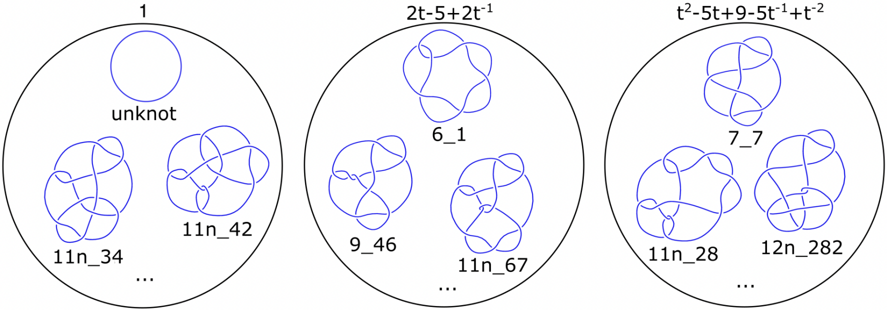
Gordian Distance and the Alexander Polynomial

Question: Does there exist a nontrivial Alexander polynomial $a(t)$ such that for any Alexander polynomial $b(t)$, there exist a pair of knots K_a and K_b which are one crossing change apart such that $\triangle_{K_a}(t) = a(t)$ and $\triangle_{K_b}(t) = b(t)$?

Answer (Kawauchi 2011, [1]): Yes! This is the case for any Alexander polynomial $a(t)$ which can be written in the form $a(t) = c(t)c(t^{-1})$ for some Laurent polynomial $c(t)$.

Gordian Distance and the Alexander Polynomial

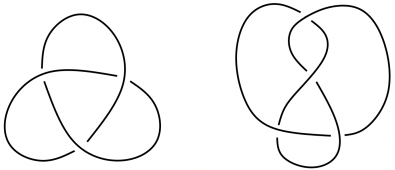
For example, $-2t+5-2t^{-1}=(2t-1)(2t^{-1}-1)$ is a slice type Alexander polynomial, so given any Alexander polynomial $q(t)$, there exists a pair of knots one crossing change apart realizing $q(t)$ and $-2t+5-2t^{-1}$.



Gordian Distance and the Alexander Polynomial

Question: Does there exist a pair of Alexander polynomials $a(t)$ and $b(t)$ such that any two knots K_a and K_b where $\triangle_{K_a}(t) = a(t)$ and $\triangle_{K_b}(t) = b(t)$ are at least two crossing changes apart?

Answer (Kawauchi 2011, [1]): Yes! For example, the Alexander polynomials of the trefoil and figure-eight knot.



$t - 1 + t^{-1}$ $\qquad$ $t - 3 + t^{-1}$

Complete Alexander Neighbors

Definition

A knot K is a **complete Alexander neighbor** if for any Alexander polynomial $p(t)$, there exists a knot K' such that K and K' are one crossing change apart and $\triangle_{K'}(t) = p(t)$.

- Kondo's result tells us that the unknot is a complete Alexander neighbor.
- Kawauchi's second result tells us that not every knot is a complete Alexander neighbor. For example, the trefoil and figure-eight knot are not complete Alexander neighbors.

Question: Does there exist a complete Alexander neighbor with nontrivial Alexander polynomial?

Obstructions to Complete Alexander Neighbor

Lemma (Nakanishi & Okada, Propositions 5 and 6 in [5])

Let K be a knot with unknotting number n and let $A_K(t)$ be an Alexander matrix of K obtained through a collection of n unknotting Dehn surgeries. Then a Laurent polynomial $p(t)$ is the Alexander polynomial of some knot K' one crossing change away from K if and only if there exist Laurent polynomials $r_1(t), ..., r_n(t)$, and $m(t)$ such that

1. $m(t) = m(t^{-1})$, $m(1) = \pm 1$, *and* $r_i(1) = 0$ *for all* $1 \leq i \leq n$, *and*
2. $$p(t) = \pm \det \begin{pmatrix} & & & r_1(t^{-1}) \\ & A_K(t) & & \vdots \\ & & & r_n(t^{-1}) \\ r_1(t) & \dots & r_n(t) & m(t) \end{pmatrix}$$

Obstructions to Complete Alexander Neighbor

Lemma (Nakanishi & Okada, Case $n = 1$ of Propositions 5 and 6 in [5])

Let K be a knot with unknotting number one. Then a Laurent polynomial $p(t)$ is the Alexander polynomial of some knot K' one crossing change away from K if and only if there exist Laurent polynomials $r(t)$ and $m(t)$ such that

1. $m(t) = m(t^{-1})$, $m(1) = \pm 1$, $r(1) = 0$ *and*
2. $p(t) = \pm \det \begin{pmatrix} \triangle_K(t) & r(t^{-1}) \\ r(t) & m(t) \end{pmatrix} = \pm m(t)\triangle_K(t) \mp r(t)r(t^{-1})$

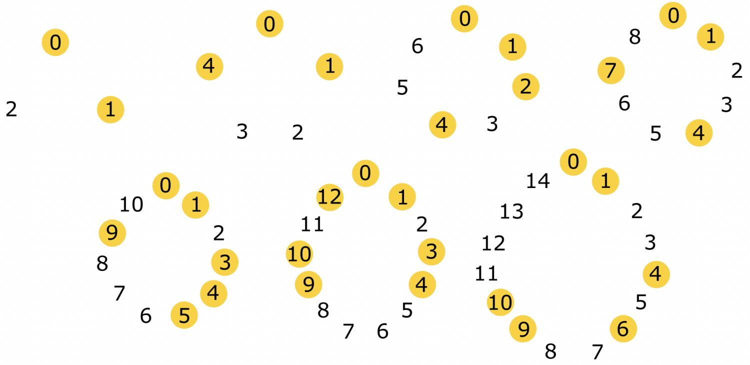
So, if K and K' are one crossing change apart and K has unknotting number one, then

$$\det(K') = \pm m(-1)\det(K) \mp (r(-1))^2.$$

Obstructions to Complete Alexander Neighbor

Definition

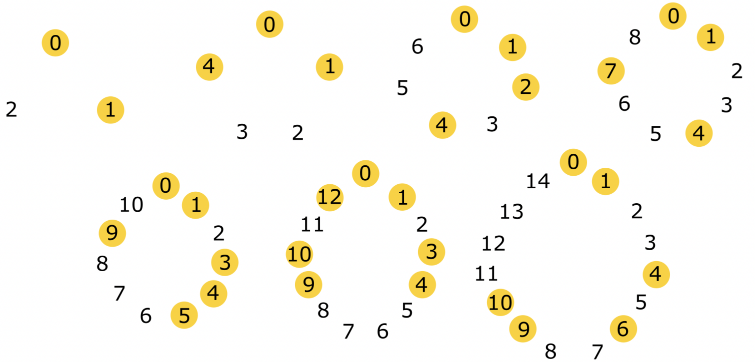
An integer q is a **quadratic residue** mod n if there exists an integer x such that $q \equiv x^2 \bmod n$.



Obstructions to Complete Alexander Neighbor

Lemma

Let $n > 1$ be an odd integer. Then n is composite or $n \equiv 1 \mod 4$ if and only if there exists some integer d such that both d and $-d$ are quadratic nonresidues mod n.



Obstructions to Complete Alexander Neighbor

Theorem (W.)

Let K be a knot with unknotting number 1, where $\det(K) > 1$ and where $\det(K)$ is composite or $\det(K) \equiv 1 \mod 4$. Then K is not a complete Alexander neighbor.

K	$\triangle_K(t)$	det(K)	unknotting number
3_1 	$t - 1 + t^{-1}$	3	1
4_1 	$t - 3 + t^{-1}$	5	1
5_1 	$t^2 - t + 1 - t^{-1} + t^{-2}$	5	2
5_2 	$2t - 3 + 2t^{-1}$	7	1
6_1 	$2t - 5 + 2t^{-1}$	9	1

Obstructions to Complete Alexander Neighbor

Proposition (Kawauchi, Corollary 4.2 from [1])

Let p be any prime number, and n, ℓ integers coprime to p. If p is an odd prime, then assume that p is coprime to $1-4n$ and that $1-4n$ is a quadratic nonresidue mod p. Consider a set of Alexander polynomials

$$\begin{aligned} S_{p,n,\ell} =& \{n(t+t^{-1})+1-2n\} \\ & \cup \{(n+\ell p^{2s+1})(t+t^{-1})+1-2(n+\ell p^{2s+1}) \mid s \in \mathbb{N}_0\} \end{aligned}$$

and let $a, b \in S_{p,n,\ell}$ such that $a \neq b$. Then for any knots K_a, K_b such that $\triangle_{K_a} = a$ and $\triangle_{K_b} = b$, we have that K_a and K_b must have Gordian distance at least two.

Obstructions to Complete Alexander Neighbor

Theorem (W.)

An Alexander polynomial of breadth 2, $q(t) = n(t + t^{-1}) + 1 - 2n$ is contained in $S_{p,n,\ell}$ for some p, n, and ℓ as defined in Corollary 4.2 from [1] if and only if $1 - 4n$ is not a square.

K	$\triangle_K(t)$	$n(t+t^{-1})+1-2n$	$1-4n$
3_1 	$t-1+t^{-1}$	$(t+t^{-1})-1$	-3
4_1 	$t-3+t^{-1}$	$-(t+t^{-1})+3$	5
5_1 	$t^2-t+1-t^{-1}+t^{-2}$		
5_2 	$2t-3+2t^{-1}$	$2(t+t^{-1})-3$	-7
6_1 	$2t-5+2t^{-1}$	$-2(t+t^{-1})+5$	9

Obstructions to Complete Alexander Neighbor

Corollary

Let K be a knot with a breadth 2 Alexander polynomial $\triangle_K(t) = n(t + t^{-1}) + 1 - 2n$. If K has unknotting number one or $1 - 4n$ is not a square, then K is not a complete Alexander neighbor.

Let K be a knot with unknotting number one and $\triangle_K(t) = n(t + t^{-1}) + 1 - 2n$. Then

$$\det(K) = \begin{cases} 1 - 4n & n \leq -1 \\ 4n - 1 & n \geq 1 \end{cases}.$$

In the case where $n \leq -1$, $\det(K) \equiv 1 \bmod 4$.
In the case where $n \geq 1$, $1 - 4n$ is not a square.

Obstructions to Complete Alexander Neighbor

Let K be a knot.

1. If K has algebraic unknotting number greater than one (which applies to 1,546 of the 2,977 prime knots with crossing number 12 or less), or
2. if K has unknotting number one and determinant which is composite or congruent to 1 mod 4 (which applies to 384 of the 2,977 prime knots with crossing number 12 or less), or
3. if K has Alexander polynomial of breadth 2 $\triangle_K(t) = n(t + t^{-1}) + 1 - 2n$ where K has unknotting number one or $1 - 4n$ is not a square (which applies to 29 of the 2,977 prime knots with crossing number 12 or less)

Then K is not a complete Alexander neighbor. All together, this eliminates 1,944 of the 2,977 prime knots with 12 crossings or fewer.

Future Directions

Recall:

Lemma (Nakanishi & Okada, Propositions 5 and 6 in [5])

Let K be a knot with unknotting number n and let $A_K(t)$ be an Alexander matrix of K obtained through a collection of n unknotting Dehn surgeries. Then a Laurent polynomial $p(t)$ is the Alexander polynomial of some knot K' one crossing change away from K if and only if there exist Laurent polynomials $r_1(t), ..., r_n(t)$, and $m(t)$ such that

1. $m(t) = m(t^{-1})$, $m(1) = \pm 1$, *and* $r_i(1) = 0$ *for all* $1 \leq i \leq n$, *and*
2. $p(t) = \pm \det \begin{pmatrix} & & & r_1(t^{-1}) \\ & A_K(t) & & \vdots \\ & & & r_n(t^{-1}) \\ r_1(t) & \dots & r_n(t) & m(t) \end{pmatrix}$

Future Directions

Conjecture

Let K be a knot with algebraic unknotting number n. Then there exists an Alexander matrix $A_K(t)$ obtained through a collection of n Dehn surgeries which transform K into a knot with trivial Alexander polynomial. Then a Laurent polynomial $p(t)$ is the Alexander polynomial of some knot K' one crossing change away from K if and only if there exist Laurent polynomials $r_1(t), \ldots, r_n(t)$, and $m(t)$ such that

1. $m(t) = m(t^{-1})$, $m(1) = \pm 1$, *and* $r_i(1) = 0$ *for all* $1 \leq i \leq n$, *and*
2. $p(t) = \pm \det \begin{pmatrix} & & & r_1(t^{-1}) \\ & A_K(t) & & \vdots \\ & & & r_n(t^{-1}) \\ r_1(t) & \ldots & r_n(t) & m(t) \end{pmatrix}$

Future Directions

Conjecture

Let K be a knot with algebraic unknotting number one, where $\det(K) \geq 3$ and where $\det(K)$ is composite or $\det(K) \equiv 1 \mod 4$. Then K is not a complete Alexander neighbor.

Conjecture

Let K be a knot whose Alexander polynomial $\triangle_K(t)$ has breadth 2. Then K is not a complete Alexander neighbor.

Obstructions From Unknotting Number One

Recall that by Nakanishi & Okada in [5], if K and K' are one crossing change apart and K has unknotting number one, then

$$\det(K') = \pm m(-1)\det(K) \mp (r(-1))^2$$

so $\det(K')$ is a quadratic residue mod $\det(K)$.

11n_162
DT Code: [6, -10, 12, 22, 16, -18, 8, 20, -4, 2, 14]
Determinant: 55

9_45
DT Code: [-6, -10, 12, 22, 16, -18, 8, 20, -4, 2, 14]
Determinant: 23

Obstructions From Unknotting Number One

Theorem (W.)

The knots $11n_{162}, 12n_{805}, 12n_{814}, 12n_{844},$ *and* $12n_{856}$ *have unknotting number greater than one.*

11n_162

12n_805

12n_814

12n_844

12n_856

Lickorish's Obstruction

Definition

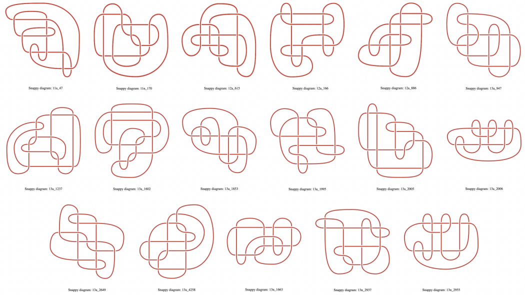
Let M be an oriented 3-manifold where $H_1(M)$ is finite. Then the **linking form** of M is $\lambda : H_1(M) \times H_1(M) \to \mathbb{Q}/\mathbb{Z}$ as defined below. Let $[\alpha], [\beta] \in H_1(M)$ represented by 1-cycles α and β in M respectively. Then $n\alpha$ bounds a disk D for some integer n. Define $\lambda([\alpha], [\beta]) = \frac{1}{n} i(D, \beta)$ where $i(D, \beta)$ is the intersection number of D and β.

Lemma (Lickorish, Lemmas 1 and 2 in [3])

If K is a knot with unknotting number one, then $S^3 \setminus K$ is obtained by $\pm\frac{\det K}{2}$-surgery on a knot in S^3 and $H_1(S^3 \setminus K)$ is cyclic with a generator g such that $\lambda(g, g) = \frac{2}{\det K} \in \mathbb{Q}/\mathbb{Z}$.

Comparing Obstructions

Of the prime knots up to 13 crossings, there are 17 examples where changing some crossing in the DT code recorded in KnotInfo [4] yields a knot one crossing change away which satisfies Nakanishi & Okada's condition on determinants to show that the unknotting number must be greater than one, but Lickorish's obstruction does not apply.



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